JOURNAL OF COMPUTATIONAL PHYSICS 5, 383–405 (1970)

A Unified Heuristic Model of Fluid Turbulence

T. H. GAWAIN

Naval Postgraduate School, Monterey, California

AND

J. W. PRITCHETT

Information Research Associates, Berkeley, California 94704

Received November 11, 1969

A unified heuristic model of fluid turbulence is proposed which supplements the basic equations of motion and continuity so as to define determinate solutions at high Reynolds number under a wide range of conditions. A single set of equations applies to all cases. Only the boundary conditions change for each application.

The Reynolds stresses are related to the mean flow heuristically, through an eddy viscosity. This also involves the turbulent energy, which is analyzed with the aid of empirical expressions for dissipation and diffusion. A characteristic local length scale is introduced, based on a generalization of von Karman's mixing length.

The model is illustrated by application to a channel and a jet. Agreement with experiment is satisfactory.

INTRODUCTION

A major obstacle to progress in the numerical integration of the Navier–Stokes equations for fluid flow problems of technical importance has been the impracticality of carrying out computations at high Reynolds number. In finite-difference calculations, for example, it may be shown that errors in the finite-difference representation of the advection terms of the momentum equation will possess a diffusion-like character and, as Reynolds number is increased, will inevitably overpower the true viscous terms and dominate the calculation. In fact, in some such schemes, the false momentum diffusion may actually become negative, causing computational instability.

Physically, the significance of this restriction is that the computational space net must be fine enough to resolve the smallest turbulent wavelengths present, while at the same time sufficiently extensive to encompass the large scale mean flow. The disparity between these two scales of size increases, of course, with increasing Reynolds number. Furthermore, even if the mean flow is known, for instance, to be steady and plane or axi-symmetric, the calculations must be three-dimensional and unsteady, since the associated turbulence is intrinsically three-dimensional and unsteady. Such a numerical approach to high Reynolds number flows is clearly out of the question, for all practical purposes.

An attractive alternative to this melancholy situation is to attempt to model the turbulent effects (Reynolds stresses) using approximate methods. Such schemes are not deducible from the equations of motion themselves, but must be formulated heuristically, using physical intuition, dimensional analysis, and the like. The test of validity of such models is simply the extent to which they reproduce experimental facts. Such an approach, of course, is not appealing to the purist, but seems to offer, at this time, the only feasible method of calculating flows of practical importance.

In incompressible fluid flow the equations of mass and momentum conservation, along with the boundary conditions, suffice in principle to establish completely the entire fluid motion. In the flow happens to be turbulent, however, the actual detailed motion becomes so complex that, although it is theoretically determinate, its actual calculation would impose an overwhelming computational burden. Furthermore, for many problems of technical importance, the results of interest are certain average properties of the flow, and the large mass of additional detailed information available is often not required, or even desired.

To circumvent this difficulty, the usual procedure is to average the equations of momentum and continuity. This results in an enormous simplification, but also, regrettably, involves a significant and irretrievable loss of essential information. Consequently, owing to the presence of the unknown Reynolds stresses created by the averaging process, the averaged equations of momentum and continuity do not in themselves comprise a determinate set.

In order to define a determinate solution, additional relations are required to fix the unknown Reynolds stresses. Unfortunately, these supplementary relations cannot be established from the original equations by any purely deductive process. For this purpose, supplementary empirical hypotheses are an unavoidable necessity. From another viewpoint, it may be stated that the averaged equations of motion show the effect of the Reynolds stresses on the mean flow. However, the reciprocal effect of the mean flow upon the Reynolds stresses is lost in the averaging process. Hence some adequate hypothesis must be found for representing this relation, at least approximately.

For this purpose, an heuristic approach which seems plausible is to postulate a relation between the Reynolds stresses and the mean flow which is analogous to the relation which is known to govern the viscous stresses. The analogue of the ordinary molecular viscosity is the so-called eddy kinematic viscosity. The problem becomes, therefore, to determine empirically the general law which governs this mean effective eddy viscosity at every space-time point in the flow field.

A number of models have been advanced over the years for obtaining the eddy viscosity distributions in various flows of technical importance. It is a striking fact that, despite the relative simplicity of the notions behind these models, they do indeed reproduce experimental data remarkably well. Unfortunately, each of these empirical theories has its own particular formulation and set of empirical constants, and there is no clear connection between the separate cases. Thus, there are more or less separate theories for turbulent flows in pipes, in ducts, in two and three dimensional boundary layers, in plane and axi-symmetric jets and wakes, and so on.

In order to formulate a more general scheme for approximating the eddy viscosity distribution in an arbitrary flow, it appears necessary to consider more parameters than are usually available from experimental results. That is, the above-mentioned models all possess one unifying characteristic—they depend for their formulation exclusively on quantities determinable directly from the mean flow. It seems intuitively clear, however, that a truly adequate model for turbulent flows should depend both on the mean flow and on the character of the accompanying turbulence. In particular, it appears reasonable that, at the very least, the local kinetic energy of the turbulent fluctuations should somehow be involved. Thus, it becomes necessary to find the space-time distribution of the turbulent energy. Fortunately, the governing energy equation can be deduced rigorously from the original equations of motion. However, the energy equation itself introduces two additional unknowns which can only be approximated in the same heuristic and empirical fashion as was the eddy viscosity. The additional unknowns are the rate of dissipation of turbulent energy into heat, and the rate of turbulent diffusion of energy.

Theory and experiment both show that the eddy viscosity, and the dissipation and diffusion functions as well, depend not only on the turbulent energy itself, but also on a local length scale parameter which can be associated with each space time point in the flow field. Von Karman was perhaps the first to point out how a physically meaningful characteristic length can be defined in terms of local space derivatives of the mean velocity field at any point in the flow. In the present paper, the original approach of von Karman is further developed and refined. It now takes into account not only the velocity derivatives at the designated point itself, but also the values in the general vicinity of the point.

By employing dimensional analysis, and by applying the available experimental data, three empirical expressions are finally obtained which determine to a reasonable approximation the eddy viscosity, the heat dissipation, and the turbulent diffusion, respectively. These expressions also involve the turbulent energy, the local length parameter, and the distance to the nearest fixed wall (if any). Of course, these empirical expressions are amenable to further investigation and development.

In this way a single consistent and determinate set of equations is established which applies in principle to any incompressible turbulent flow field. Only the boundary conditions differ for each specific application.

As an interesting historical note, it seems worthwhile to mention that the heuristic model developed in this paper resembles in some respects the approach first suggested by Prandtl [1] in 1945. It also has some similarities to, and some differences from, the more recent work of Harlow, Nakayama and Hirt ([2], [3]).

Development of the Model

The equations of motion for an incompressible, viscous fluid may be written:

$$\partial U_j'/\partial x_j = 0 \qquad \text{(continuity)} \tag{1}$$

$$\frac{\partial}{\partial t}(U_i') + \frac{\partial}{\partial x_j}(U_i'U_j') = \nu \frac{\partial^2 U_i'}{\partial x_j\,\partial x_j} - \frac{\partial \Phi'}{\partial x_i} \qquad \text{(momentum)} \tag{2}$$

where U_1', U_2', U_3' are velocity components in the directions x_1, x_2, x_3; ν is kinematic viscosity, and Φ' is kinematic pressure (pressure/density). The velocity components and the pressure may be separated into mean and fluctuating parts:

$$U_i' = U_i + u_i$$
$$\Phi' = \Phi + \varphi.$$

These may be inserted into the equations of motion, and the results ensemble-averaged. Further manipulation leads to three equations of interest:

$$\frac{\partial}{\partial x_j}(U_j) = 0 \quad \text{(mean flow continuity)} \tag{3}$$

$$\frac{\partial}{\partial t}(U_i) + \frac{\partial}{\partial x_j}(U_iU_j) = \frac{\partial}{\partial x_j}\left[\nu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \overline{u_iu_j}\right] - \frac{\partial \Phi}{\partial x_i}$$
$$\text{(mean flow momentum)} \tag{4}$$

$$\frac{\partial}{\partial t}\left(\frac{\overline{u_ju_j}}{2}\right) + \frac{\partial}{\partial x_k}\left(U_k\frac{\overline{u_ju_j}}{2}\right) = -\frac{\overline{u_ju_k}}{2}\left(\frac{\partial U_j}{\partial x_k} + \frac{\partial U_k}{\partial x_j}\right)$$
$$-\frac{\nu}{2}\overline{\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)} \tag{5}$$
$$-\frac{\partial}{\partial x_k}\overline{\left[u_k\left(\frac{u_ju_j}{2} + \varphi\right)\right]}$$
$$+\nu\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_k}\left(\frac{\overline{u_ju_j}}{2}\right) + \frac{\partial}{\partial x_j}(\overline{u_ju_k})\right]$$
$$\text{(turbulent energy).}$$

The terms on the right of the energy equation (5) represent, respectively, turbulent energy production corresponding to the work done by the mean flow against the Reynolds stresses, dissipation of turbulent energy to heat, turbulent diffusion of energy, and molecular diffusion. For problems at high Reynolds number, the last term is vanishingly small; it will be hereafter ignored.

We postulate that the Reynolds stresses can be adequately related to the strain rates of the mean flow through the law:

$$-\overline{u_i u_j} = -\tfrac{1}{3}\overline{u_k u_k}\delta_{ij} + \epsilon(\partial U_j/\partial x_i + \partial U_i/\partial x_j), \tag{6}$$

where $\delta_{ij} = 0$ for $i \neq j$, and $= 1$ for $i = j$; ϵ is the so-called eddy kinematic viscosity. If we drop the last term in Eq. (5) and insert postulate (6), we obtain:

$$\frac{\partial}{\partial t}(U_i) + \frac{\partial}{\partial x_j}(U_i U_j) = \frac{\partial}{\partial x_j}\left[(\nu + \epsilon)\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] - \frac{\partial P}{\partial x_i}$$

(mean flow momentum) (7)

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_k}(U_k E) = \epsilon\Omega^2 - \frac{\nu}{2}\overline{\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)} - \frac{\partial}{\partial x_k}\left[\overline{u_k\left(\frac{u_j u_j}{2} + \varphi\right)}\right] \tag{8}$$

(turbulent energy),

where

$$\Gamma_{jk} = (\partial U_j/\partial x_k + \partial U_k/\partial x_j) \quad \text{(mean flow strain rate)} \tag{9}$$

$$\Omega^2 = \tfrac{1}{2}\Gamma_{jk}\Gamma_{jk} \tag{10}$$

$$E = \overline{u_j u_j}/2 \quad \text{(mean turbulent kinetic energy)} \tag{11}$$

$$P = \Phi + \tfrac{2}{3}E \quad \text{(total pressure).} \tag{12}$$

Now consider the last term of Eq. (8). It is useful for purposes of discussion to break this into its two constitutent terms:

$$\overline{u_k\left(\frac{u_j u_j}{2} + \varphi\right)} = \overline{u_k\frac{u_j u_j}{2}} + \overline{u_k\varphi}. \tag{13}$$

The first of these terms clearly represents the net advective transport of the fluctuating turbulent kinetic energy $u_j u_j/2$ by the action of the turbulent velocity fluctuations u_k. Clearly, the turbulent kinetic energy is both a scalar and an extensive property. It may be regarded as something which is physically transported along with the element of fluid mass. In this respect it is quite analogous to other extensive properties such as heat or salinity. It is customary to represent such net turbulent transport by an appropriate empirical diffusion coefficient, as we have already done for momentum with the kinematic eddy viscosity. Hence we write

$$\overline{u_k\left(\frac{u_ju_j}{2}\right)} = -\epsilon' \frac{\partial}{\partial x_k}\left(\frac{\overline{u_ju_j}}{2}\right) = -\epsilon'\left(\frac{\partial E}{\partial x_k}\right). \tag{14}$$

On the other hand, the pressure-velocity correlation term $\overline{u_k\varphi}$ is in a somewhat different category owing to the fact that the pressure φ is an intensive property. It is not an extensive property which can be regarded as being transported along with the fluid mass. We note, however, that φ is a dependent variable. This means that whenever the spatial distribution of the velocity fluctuations u_k is specified, the corresponding spatial distribution of φ is likewise fixed. Hence, looking at the matter statistically, we can say that φ is correlated in some fashion with u_k. In fact, taking dimensional considerations into account, we can improve this statement and say that φ is statistically correlated with $(u_ju_j/2)$. For example, in the limiting case of an inviscid fluid, Bernoulli's equation informs us that regions of higher than average kinetic energy will tend to be regions of lower than average pressure, and vice versa. Consequently, we are justified in writing the pressure term in the analogous form

$$\overline{u_k\varphi} = -\epsilon''(\partial E/\partial x_k) \tag{15}$$

so that the overall effect becomes

$$\overline{u_k(u_ju_j/2 + \varphi)} = -(\epsilon' + \epsilon'')\,\partial E/\partial x_k\,. \tag{16}$$

Normally, ϵ' is a positive quantity. However, because of the generally negative correlation between pressure and kinetic energy, we expect ϵ'' to be usually negative and smaller in magnitude than ϵ'. Hence $(\epsilon' + \epsilon'')$ should be positive but smaller than ϵ' alone. Moreover, ϵ' and ϵ'' are both of the same dimension as the kinematic eddy viscosity ϵ. For these reasons, it is advantageous to write

$$(\epsilon' + \epsilon'') = \gamma\epsilon, \tag{17}$$

where γ is a dimensionless coefficient roughly of order unity. Hence, we finally write:

$$\overline{u_k(u_ju_j/2 + \varphi)} = -\gamma\epsilon(\partial E/\partial x_k) \tag{18}$$

The kinematic eddy viscosity ϵ itself, of course, remains to be determined. Unfortunately, ϵ is not a simple property of the fluid, but is rather a complex property of the turbulent flow field and its interaction with the mean flow. In order to formulate a plausible model for ϵ, it is first necessary to consider the establishment of *scales of size* appropriate to the description of the mean flow and/or the turbulent field.

It is possible formally to establish an appropriate scale of turbulence (or *macroscale*) at a space-time point $(\vec{x}, t)$ in terms of the statistical properties of the turbulence in the vicinity of the point. Let the correlation tensor at the point be defined as follows:

$$R_{ij}(\vec{x}, \Delta\vec{x}) = \frac{\overline{u_i(\vec{x} + \Delta\vec{x})\, u_j(\vec{x} - \Delta\vec{x})}}{\overline{u_k(\vec{x})\, u_k(\vec{x})}}. \tag{19}$$

The first invariant of this tensor is:

$$R_{ii}(\vec{x}, \Delta\vec{x}) = \frac{\overline{u_i(\vec{x} + \Delta\vec{x})\, u_i(\vec{x} - \Delta\vec{x})}}{\overline{u_k(\vec{x})\, u_k(\vec{x})}}. \tag{20}$$

Now the desired characteristic length λ^* may be defined as follows:

$$\lambda^{*2}(\vec{x}) = \frac{\int |R_{ii}| (\Delta\vec{x} \cdot \Delta\vec{x})\, dv'}{\int |R_{ii}|\, dv'} \tag{21}$$

where $dv' = dx_1'\, dx_2'\, dx_3'$ represents an infinitesimal volume element at the variable point $\vec{x} + \Delta\vec{x}$, and

$$\Delta\vec{x} \cdot \Delta\vec{x} = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2.$$

The integrals in (21) extend over all space. Of course, these integrals are finite, despite the infinite domains of integration, because of the rapid decay in the correlation function at large separations.

The numerator of Eq. (21) represents the second moment of the correlation function. It is therefore analogous to a moment of inertia. Also, λ^{*2} is analogous to a variance in probability theory. It is well-known that such second moments attain their minimum values when evaluated with respect to the centroidal point of the distribution in question. In the present instance, the correlation function is symmetrical with centroid at $\Delta\vec{x} = 0$, and hence λ^{*2} represents the minimum possible varianec of the correlation function. These properties make λ^* a particularly appropriate measure of the length scale of turbulence in the vicinity of the point $(\vec{x}, t)$.

As has been seen, it is convenient to resolve turbulent fluid flow into two distinctly complementary aspects which we then refer to as *the mean flow* and *the turbulent fluctuations*. While such a distinction is useful for conceptual and computational purposes, it should not be forgotten that these two aspects are inextricably coupled within a single unified process. This fact provides a useful hint in connection with the problem of formulating appropriate auxiliary hypotheses to supplement the averaged equations of motion. Such hypotheses will naturally be formulated in terms of the fundamental physical parameters which characterize the overall turbulence in the vicinity of the point $(\vec{x}, t)$. Among these quantities are the turbulent kinetic energy $E(\vec{x}, t)$ and the macroscale of turbulence $\lambda^*(\vec{x}, t)$.

Of course, in most problems of practical importance, interest centers chiefly on the mean flow field. More often than not, therefore, we do not have available the data which would be needed to establish such a largely unseen characteristic of the turbulence as the macroscale λ^*. Consequently, it is of doubtful utility to formulate any heuristic theory of turbulence directly in terms of such quantities. Thus, we must at this stage of our knowledge replace the fundamental macroscale λ^* by another quantity which is more or less equivalent, but which is far more amenable to observation and calculation.

Fortunately, for this purpose we can take advantage of the intimate connection between the local turbulence field and the local mean flow field noted above. Presumably because of this connection the local length scale must somehow be reflected in the local characteristics of the mean flow, and must therefore be deducible from the mean flow.

While the exact nature of this relationship between the macroscale and the mean flow is far from clear, we can legitimately infer that the relationship is inherently somewhat local in character. This follows from a fact universally revealed by all observed correlation data: the correlation curves always show a rapid decrease in correlation with increasing separation. Moreover, the correlations become negligible at separations larger than about two or three multiples of λ^*.

Consequently, we hypothesize that a quantity more or less equivalent to the true macroscale λ^* can be defined and computed in terms of the observable characteristics of the local mean flow. Let us call this quantity the *apparent macroscale* and designate it by the symbol λ. It follows from the foregoing discussion that the definition of λ in the vicinity of an arbitrary point $(\vec{x}, t)$ should depend only on the mean flow conditions in a finite region surrounding that point. This principle is more realistic than any method which seeks to define this macroscale in terms of mean flow quantities only at $(\vec{x}, t)$ itself. On the other hand, it avoids the opposite extreme which would hold that λ is somehow dependent on all points in the flow field, no matter how remote. Naturally, the definition of λ should emphasize mean flow conditions in the immediate neighborhood of the point $(\vec{x}, t)$, should give progressively less weight to conditions farther away, and should finally neglect mean flow effects at points sufficiently remote. This requirement suggests the general idea of defining λ in terms of appropriate weighted averages of certain mean flow quantities.

The preferred choice of weighting function for this purpose would be the correlation function R_{ii} itself, if it were known. Of course, if R_{ii} were known, it could be used to find the true macroscale λ^* directly, and there would be no need for a weighting function. However, it is not known, and we are therefore obliged to use an heuristic substitute, preferably a function which resembles the correlation function to a certain extent. Fortunately, great accuracy is not required in this regard, for weighted averages tend to be relatively insensitive to minor variations in the form of weighting function used.

In view of these considerations, we choose the Gaussian curve as an appropriate form of heuristic weighting function. It expresses the general trends of correlation in a suitable manner, and has various convenient mathematical properties as well. Consequently, we finally write the weighting functions in normalized form as follows:

$$w(\vec{x}, \Delta\vec{x}) = \frac{\exp\left(-\dfrac{\Delta\vec{x}\cdot\Delta\vec{x}}{\lambda^2(\vec{x})}\right)}{\displaystyle\int_{\text{all space}} \exp\left(-\dfrac{\Delta\vec{x}\cdot\Delta\vec{x}}{\lambda^2(\vec{x})}\right) dv'} . \tag{22}$$

We have been considering any arbitrary and general turbulent flow field of which the mean flow may be either steady or unsteady. With every point of the field there is associated the parameter λ which expresses a length scale characteristic of the mean flow pattern in the vicinity of the point. Our problem is to devise a suitable explicit definition of this parameter in a way which meets various essential requirements. Some of these have already been discussed. There are additional requirements as well, among which are the following:

(a) λ must be a true scalar, and therefore invariant with respect to any rotation, reflection, translation, or acceleration of the reference axes.

(b) λ should also be everywhere continuous, finite and positive (except possibly in certain singular regions, such as at a solid boundary).

(c) If feasible, the mean flow quantity λ^2 should preferably be related to the correlation quantity λ^{*2} such that the ratio of the two is close to unity, or at least remains as nearly constant as possible.

These requirements can be satisfied as follows. Let the strain rate tensor be defined as:

$$\Gamma_{ij} = \frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \qquad \text{(see Eq. 9)} \tag{23}$$

Note that for the incompressible case which we are considering, the first invariant vanishes:

$$\Gamma_{ii} = 2(\partial U_i/\partial x_i) = 0 \qquad \text{by continuity.} \tag{24}$$

Therefore, there is no net dilatation, and the Γ_{ij} represent purely distortional effects.

We now define a generalized strain rate Ω, and a generalized strain rate gradient Ω' as follows:

$$\Omega^2 = \tfrac{1}{2}\Gamma_{ij}\Gamma_{ij} \qquad \text{(see Eq. 10)} \tag{25}$$

$$\Omega'^2 = (\partial\Omega/\partial x_i)(\partial\Omega/\partial x_i). \tag{26}$$

From these definitions, the following useful quantity can be obtained:

$$(\Omega\Omega')^2 = \tfrac{1}{4}(\partial\Omega^2/\partial x_i)(\partial\Omega^2/\partial x_i). \tag{27}$$

It can be shown that while U_i and Γ_{ij} depend on the orientation of the reference axes, the quantity Ω^2 is invariant in this regard. The same is true of $(\Omega\Omega')^2$. Hence λ^2 can be conveniently defined in terms of these variables.

In accordance with the above notions, we therefore define as follows:

$$\lambda^2(\vec{x}) = I^2(\vec{x})/J^2(\vec{x}), \tag{28}$$

where

$$I^2(\vec{x}) = \int_{\text{all space}} w(\vec{x}, \vec{x}')\, \Omega^4(\vec{x}')\, dv' \tag{29}$$

$$J^2(\vec{x}) = \int_{\text{all space}} w(\vec{x}, \vec{x}')(\Omega\Omega'(\vec{x}'))^2\, dv' \tag{30}$$

and

$$w(\vec{x}, \vec{x}') = \frac{\exp\left(-\dfrac{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}{\lambda^2(\vec{x})}\right)}{\displaystyle\int_{\text{all space}} \exp\left(-\dfrac{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}{\lambda^2(\vec{x})}\right) dv'}. \tag{31}$$

It should be noted that some difficulty will be experienced in calculating the λ distribution directly from an arbitrary velocity distribution, since λ appears on both sides of the defining equation. Methods used to approximate λ will be discussed later.

Having defined an appropriate macroscale, we now postulate a formulation for the eddy viscosity ϵ using physical and dimensional reasoning as follows:

$$\frac{\epsilon}{\lambda\sqrt{2E}} = \alpha. \tag{32}$$

The dimensionless coefficient α is a slowly varying universal function not predictable from theory, but determinable from experimental data. As such it is in the same class as the function γ mentioned earlier in connection with turbulent diffusion (see Eq. 16). Clearly, of course, the usefulness of formulation (32) depends on the extent to which it reproduces observable data, that is, the extent to which α is a predictable and well-behaved function.

Now consider the dissipation of turbulent energy into heat (second term on the right of Eq. 8). Dimensional considerations suggest that this term can be expressed to good advantage in the form

$$\dot{E}_H = \frac{\nu}{2}\overline{\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)} = \frac{\nu(2E)}{\lambda_D^2}\,. \tag{33}$$

This expression amounts to a definition of the so-called dissipation length λ_D which is a length characteristic of the energy dissipating wave lengths of the turbulent spectrum. This form of expression would be fairly convenient if the dissipation length λ_D happened to be related to the macroscale λ in some fairly simple and invariant way. Unfortunately, this does not seem to be the case. Experimental studies such as those by Laufer clearly suggest that at sufficiently high Reynolds number the heat dissipation effects tend to become independent of Reynolds number. This implies that, contrary to Eq. (33), the optimal form of expression for $\dot{E}_H$ is one which does not contain viscosity ν explicitly. This curious fact can be given a reasonable interpretation by considering turbulent energy processes from a spectral point of view. It is well-known that the work input occurs mainly at the long wave length end of the spectrum, and that dissipation to heat takes place at much shorter wave lengths. At high Reynolds numbers, there is a broad wave length range in between, the so-called inertial range, for which both work input and heat dissipation are negligible. Each wave length within this inertial range simply receives energy from longer wave length components and transmits this energy to shorter wave length components. It would appear that the rate of transmission of energy through the spectrum is largely controlled by this process in the inertial range, rather than by the viscous dissipation process itself. Apparently the viscous dissipation rate easily adjusts itself as may be required to dissipate all the energy coming through the inertial range. Since the viscous process is not rate controlling, the effect of viscosity tends to drop out of the experimental picture.

These notions, combined with a considerable amount of numerical experimentation, suggest the following formulation for the dissipation length. Consider two lengths L_1 and L_2 defined as follows:

$$L_1^2 = \nu^2/2E \tag{34}$$

$$L_2^3 = 2E/J, \tag{35}$$

where J is defined in Eq. (30). We now form the relation

$$L_1L_2/\lambda_D^2 = \beta \tag{36}$$

and postulate that β is another slowly varying universal function of the same sort as α and γ. The energy dissipation term now becomes

$$\dot{E}_H = \beta(2E)^{7/6}\,J^{1/3} \tag{37}$$

which shows the required independence of molecular kinematic viscosity. Actually, the above formulation does seem to match experimental data fairly well. It should be pointed out, however, that experimental information concerning this term is very sparse, and much more data is needed in this regard.

The rate of turbulent energy dissipation in pipe flow at a Reynolds number of 500,000 has been measured by Laufer [4]. His results are shown in Fig. 1. Note that Laufer regarded his measured dissipation data as too low and attempted to estimate better values as indicated. The energy dissipation calculated from Eq. (37) using Laufer's measured energy distribution and assuming the Nikuradse velocity profile is shown as the solid line in Fig. 1.

Thus, the complete model has now replaced the original momentum and energy equations (4) and (5) with heuristic substitutes

$$\frac{\partial}{\partial t}(U_i) + \frac{\partial}{\partial x_j}(U_i U_j) = \frac{\partial}{\partial x_j}\left[(\nu + \alpha\lambda\sqrt{2E})\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right] - \frac{\partial P}{\partial x_i} \quad \text{(mean flow momentum)} \tag{38}$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(U_j E) = \alpha\lambda\sqrt{2E}\,\Omega^2 - \beta(2E)^{7/6} J^{1/3} + \frac{\partial}{\partial x_j}\left(\alpha\gamma\lambda\sqrt{2E}\,\frac{\partial E}{\partial x_j}\right) \quad \text{(turbulent energy).} \tag{39}$$

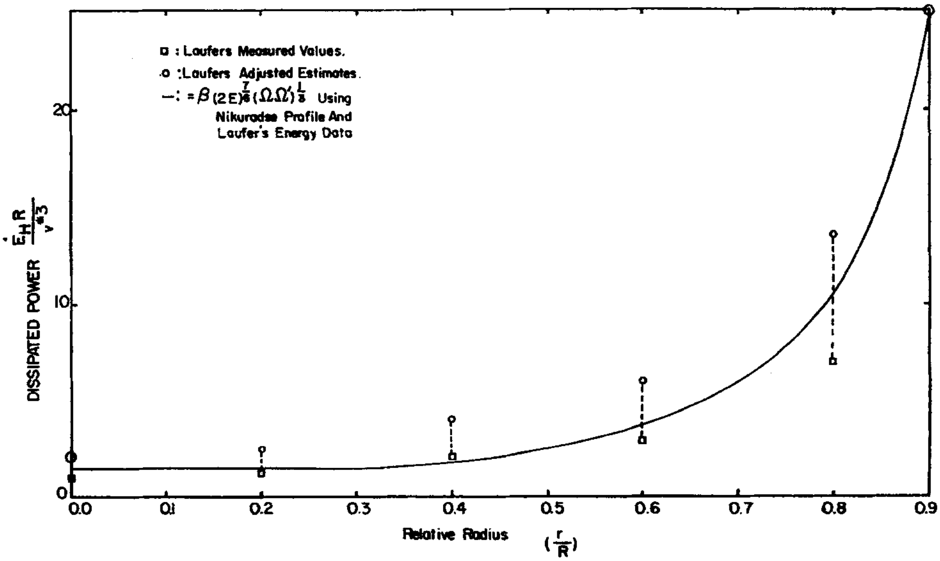

FIG. 1. Turbulent energy dissipation rate in a pipe.

In practice, the usefulness of the above formulation hinges on whether the dimensionless coefficients α, β, and γ behave in a reasonably stable and simple manner. Our present indications are that this is indeed so. On the basis of work done so far, the following expressions seem to provide satisfactory agreement with the available data:

$$\alpha = 0.065 \left\{1 + \exp\left[-\left(\frac{y}{\lambda} - 1\right)^2\right]\right\} \tag{40}$$

$$\frac{1}{\beta} = 3.7 \left\{1 + \exp\left[-\left(\frac{y}{\lambda} - 1\right)^2\right]\right\} \tag{41}$$

$$\gamma = 1.4 - 0.4 \exp\left[-\left(\frac{y}{\lambda} - 1\right)^2\right]. \tag{42}$$

In these expressions, y is the distance to the nearest fixed boundary. As the boundary is approached y approaches zero, but it turns out that under these conditions the length scale parameter λ also approaches zero simultaneously, so that the ratio (y/λ) remains finite. In fact, it happens that at the wall itself, (y/λ) equals unity (not zero). In any case, it is stipulated in connection with the above formulae that (y/λ) shall be arbitrarily assigned a lower limit of unity. Thus, the maximum possible range of variation of the above three functions is not large. On the other hand, far from any fixed boundary, or in the absence of fixed boundaries, we set $y = \infty$ whereupon α, β, and γ reduce to three simple constants.

As was mentioned earlier, the macroscale λ and the quantity J, although in principle determined by Eq. (28) through (31), are in practice difficult to calculate in that form. In order to make useful calculations, it is necessary to resort to an interation process such as the following. Let the $(n + 1)$-th approximation to λ be defined as:

$$\lambda_{n+1}^2(\vec{x}) = I_{n+1}^2(\vec{x})/J_{n+1}^2(\vec{x}) \tag{43}$$

where

$$I_{n+1}^2(\vec{x}) = \int_{\text{all space}} w_n(\vec{x}, \vec{x}')\, \Omega^4(\vec{x}')\, dv' \tag{44}$$

$$J_{n+1}^2(\vec{x}) = \int_{\text{all space}} w_n(\vec{x}, \vec{x}')[\Omega\Omega'(\vec{x}')]^2\, dv'. \tag{45}$$

The weighting function w_n is defined as follows. Let

$$W_n(\vec{x}, \vec{x}') = \exp\left(-\frac{(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{\lambda_n^2(\vec{x})}\right). \tag{46}$$

Then

$$w_n(\vec{x}, \vec{x}') = \frac{W_n(\vec{x}, \vec{x}')}{\int_{\text{all space}} W_n(\vec{x}, \vec{x}')\, dv'}. \tag{47}$$

We now define (in principle)

$$\lambda^2(\vec{x}) = \lim_{n\to\infty} \lambda_n^2(\vec{x}). \tag{48}$$

Advantage should be taken, of course, of any knowledge concerning the flow to hasten the convergence of the λ_n's. In particular, if the flow is self-similar, such as the flow in a pipe, duct, or boundary layer, or the far downstream region of a turbulent jet or wake, calculations need only be performed on a single representative cross-section of the flow, and the results rescaled according to the appropriate nondimensionalizing parameters.

As an initial guess to start the iteration process, we take $\lambda_0 = \infty$ everywhere. Then in computing λ_1, we find that the weighting function W_0 is equal to unity everywhere, and so we obtain simply

$$\lambda_1^2 = \frac{\int_{\text{all space}} \Omega^4 \, dv}{\int_{\text{all space}} (\Omega\Omega')^2 \, dv} \tag{49}$$

a constant independent of position. Thus, λ_2 is the first nonconstant approximation to λ that we obtain.

Numerical experiments with various flow fields have shown that, in general, the convergence of the λ_n's is extremely rapid. This is illustrated in Table I which shows

TABLE I

Convergence of Macroscale Distribution in a Two-Dimensional Channel with Parabolic (Laminar) Velocity Profile[a]

y/b	λ_n/b				
	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 20$
0.0	∞	0.775	0.695	0.674	0.664
0.2	∞	0.775	0.706	0.693	0.689
0.4	∞	0.775	0.731	0.727	0.727
0.6	∞	0.775	0.758	0.758	0.758
0.8	∞	0.775	0.782	0.781	0.781
1.0	∞	0.775	0.802	0.800	0.800

[a] $2b$ = channel width y = distance from centerline.

successive numerical approximations to the λ distribution in laminar flow in a two-dimensional channel. Consequently, for the purposes of the calculations presented in this paper, the approximation was made:

$$\lambda(\vec{x}) \approx \lambda_2(\vec{x}). \tag{50}$$

Some improvement could possibly be gained by continuing to higher approximations, but in view of the gross simplifications already inherent in the model as a whole, the authors feel that such refinement is not warranted. Moreover, absolute accuracy in this regard is not essential. It is necessary, however, that the chosen order of approximation be adhered to consistently. It is also important to adjust the initially undetermined numerical constants of the model so that the chosen order of approximation gives satisfactory agreement with experimental data. The significant numerical constants are those that occur in the heuristic expressions for the quantities α, β, and γ.

Numerical Results

Two widely different cases of turbulent flow have so far been calculated explicitly using the heuristic model described in the preceeding section. These are the flow in a two-dimensional channel (dominated by boundary layer effects) and in the far–downstream region of an axisymmetric jet (a case of free turbulent flow). Both cases are, of course, steady flows. The model may also be applied to unsteady flows, but the purpose of these calculations was to ascertain the agreement (if any) between calculated results and experimental facts, and there is no adequate available experimental data on unsteady turbulent flows.

In principle, the heuristic model may be employed in an unsteady sense and the calculation allowed to proceed until a steady state is reached. In practice, however, this is an expensive approach in terms of computation. The method used in the present calculations was as follows:

(1) The velocity distribution was taken from experimental results. Using the momentum equation, it was then possible to calculate the Reynolds stress distribution required to maintain the observed velocity distribution. (Experimental measures of the Reynolds stresses are also available—a comparison of the two yields an estimate of the reliability of the experimental data.)

(2) Following the steps summarized in Eqs. (43) through (50), the macroscale (λ) and J distributions were calculated, assuming the experimental velocity distribution.

(3) The turbulent energy (E) at equilibrium was then computed numerically using Eq. (39), by dropping the unsteady term and imposing appropriate boundary conditions.

(4) Using the computed energy distribution, the computed macroscale distribution and the observed velocity field, a Reynolds stress distribution was then calculated (see Eqs. 6 and 32).

(5) The Reynolds stress distribution thus computed from the heuristic model was compared with the *required* distribution found from the momentum analysis in

step (1) above. The energy distribution calculated in step (3) was also compared with experimental measurements.

Consider the steady, incompressible turbulent flow through a two-dimensional channel of uniform height 2b as shown in Fig. 2. Let y be the distance from the

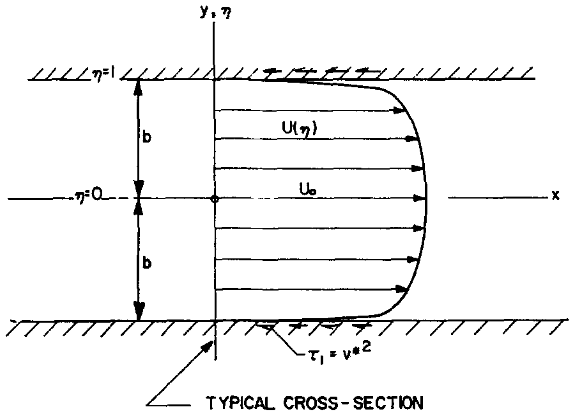

FIG. 2. Flow in a two dimensional channel.

channel centerline. Choosing b as the reference unit of length, the corresponding dimensionless distance becomes

$$y/b = \eta. \tag{51}$$

Let the kinematic shear stress at the wall be

$$\tau_w = v^{*2} \tag{52}$$

[redacted]

The kinematic shear stress τ varies linearly across the channel so that at any location η we have

$$\tau = \eta. \tag{54}$$

According to our present notation, if U is the mean velocity at any η, then

$$\partial U/\partial\eta = \Omega \tag{55}$$

$$\partial^2 U/\partial\eta^2 = \Omega'. \tag{56}$$

We will take the von Karman velocity profile for this case as representative of experimental results:

$$U = \frac{1}{0.36}(\ln(1 - \sqrt{\eta}) + \sqrt{\eta}). \tag{57}$$

Following the prescribed formulae, it may be shown that for this case $\lambda_1 = 0$. Actually, this is only true in the limit of infinite Reynolds number. For finite Reynolds number, there exists a small region near the wall (the *laminar sublayer*) in which the velocity profile is linear. In the region very near the wall, the actual velocity profile may be written (neglecting additive constants):

$$\begin{aligned} U &= \Omega_0(1 - \eta) && \text{for} \quad (1 - \eta) < \delta \\ &= \Omega_0\delta(1 + \ln(1 - \eta)) && \text{for} \quad \delta \leqslant (1 - \eta) \ll 1. \end{aligned} \tag{58}$$

In this instance we find that

$$\lambda_1 = 2.58\delta, \tag{59}$$

that is, λ_1 is of the same order of size as the thickness of the laminar sublayer. It is an interesting fact that, if λ_1 is taken as zero, we obtain

$$\lambda_2^2 = \Omega^2/\Omega'^2 \approx \lambda^2, \tag{60}$$

that is, our characteristic length is proportional to the classical von Karman mixing length.

With the functions Ω, λ, J, α, β, and γ now known, we can proceed to solve the energy equation. The unsteady term and the convection term both vanish for the present case, so we obtain

$$\alpha\lambda\sqrt{2E}\,\Omega^2 - \beta(2E)^{7/6}J^{1/3} + \frac{\partial}{\partial\eta}\left(\alpha\gamma\lambda\sqrt{2E}\frac{\partial E}{\partial\eta}\right) = 0 \tag{61}$$

wherein the turbulent energy E is the only unknown. The appropriate boundary conditions are:

$$\partial E/\partial\eta = 0 \quad \text{at} \quad \eta = 0 \tag{62}$$

$$E = 0 \quad \text{at} \quad \eta = 1. \tag{63}$$

The above equation for energy was expanded, expressed in finite-difference form, and integrated numerically using 100 stations equally spaced from $\eta = 0$ to $\eta = 1$. The resulting turbulent energy distribution is shown in Fig. 3 as the solid line. It may be seen that the curve obtained in this way is in reasonably good agreement with the experimental data of Reichardt [5] and of Laufer [6].

Once the energy distribution is known, the dimensionless Reynolds stress may be computed from the relation

$$-\overline{uv} = \alpha\lambda\sqrt{2E}\,\Omega. \tag{64}$$

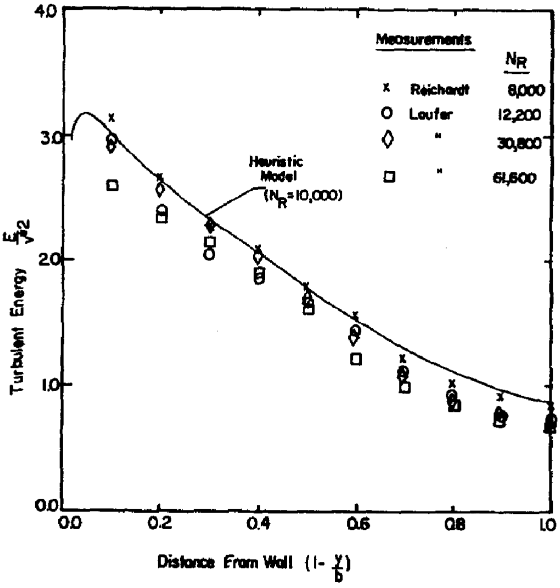

FIG. 3. Turbulent energy distribution in a two dimensional channel.

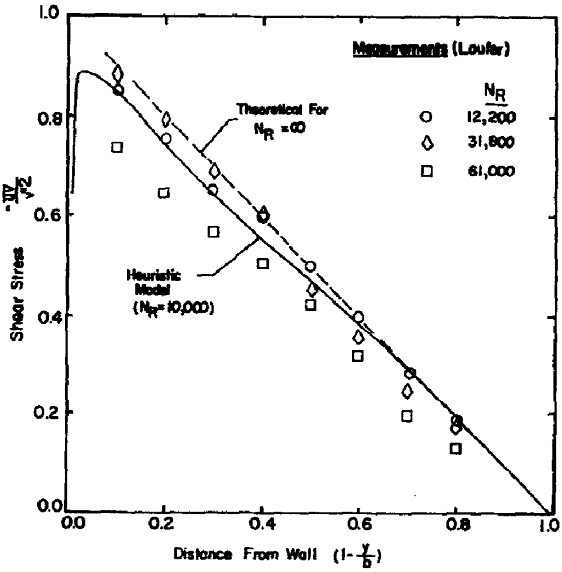

FIG. 4. Reynolds shear stress distribution in a two-dimensional channel.

Except very near the wall, the above Reynolds stress should agree with the total shear stress defined by Eq. (54).

In Fig. 4, the solid line shows the above heuristic estimate of the Reynolds stress, computed for a Reynolds number of 10,000. The dashed straight line represents the true total stress, which should agree closely with the Reynolds stress everywhere except in the immediate vicinity of the wall. The agreement is seen to be good; the computed results are on the whole more accurate than are Laufer's experimental measurements shown on the figure. We conclude that this example tends to substantiate the proposed heuristic model of turbulence.

Now consider a free axi-symmetric turbulent jet discharging into a quiescent atmosphere as shown in Fig. 5. The radial and axial coordinates are r and z and the corresponding mean velocity components are U and V, respectively.

Experimental results indicate that the velocity profiles of the mean flow at various cross-sections are self-similar. That is, if z is measured from a suitable virtual origin as shown, the stream function Ψ can be reduced to the form

$$\Psi = U_0 bzF(\eta), \tag{65}$$

where $\eta = (r/z)$ and where U_0 and b are constants of the jet.

Consequently, the velocity components of the mean flow become

$$U = -\frac{1}{r}\left(\frac{\partial\Psi}{\partial z}\right)_r = \frac{U_0 b}{z}\left(F' - \frac{F}{\eta}\right) \tag{66}$$

$$V = \frac{1}{r}\left(\frac{\partial\Psi}{\partial r}\right)_z = \frac{U_0 b}{z}\frac{F'}{\eta} \tag{67}$$

where $F' = (\partial F/\partial\eta)\, z$. (68)

The known experimental results also indicate that the generalized velocity distribution through the jet is very well approximated by the simple expression

$$\frac{F'}{\eta} = f = e^{-\eta^2/s^2}, \tag{69}$$

where s is a characteristic constant for a turbulent jet and has the known value 0.102.

Furthermore, it is useful and convenient to adopt units of length and time such that when expressed in these units $U_0 = 1$ and $b = 1$. Hence, symbols like ϵ, P, u, v, and so on will now represent the corresponding dimensionless versions of kinematic eddy viscosity, kinematic pressure, velocity components, and the like.

The next step is to transform Eq. (7) (the momentum equation) into the z, η coordinate system defined above. The foregoing dimensionless functions ϵ and P may be introduced therein, and the results reduced. If the pressure gradients in the

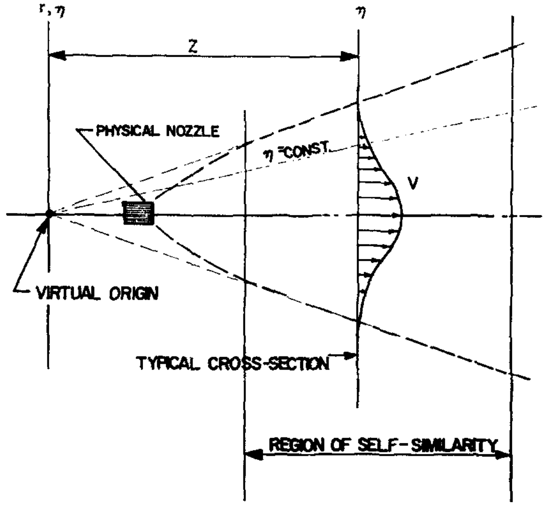

FIG. 5. The axi-symmetric jet.

jet are neglected (which experimental information indicates is a safe assumption), an equation for the distribution of ϵ results:

$$\frac{\partial \epsilon}{\partial \eta} + \left[\frac{\left(2 - \frac{4}{s^2}\right) - \left(s - \frac{2}{s^2}\right)\frac{2\eta^2}{s^2} + \frac{4\eta^4}{s^4}}{\left(1 - \frac{2}{s^2} + \frac{2\eta^2}{s^2}\right)\eta} \right] \epsilon = \frac{(1 - 2f)}{\left(1 - \frac{2}{s^2} + \frac{2\eta^2}{s^2}\right)\eta}. \tag{70}$$

This equation may be readily integrated numerically. With the dimensionless eddy viscosity now known, it becomes a simple matter to find the corresponding dimensionless Reynolds shear stress:

$$-\overline{uv} = \epsilon[\partial U/\partial z + \partial V/\partial r]. \tag{71}$$

Using equations (66) through (68) for the velocity components, we may then calculate the dimensionless shear stress numerically. It is shown as the dashed line in Fig. 7.

To form the mean flow parameters required for the heuristic model, we note that, in cylindrical coordinates,

$$\Omega^2 = 2\left(\frac{\partial U}{\partial r}\right)^2 + 2\left(\frac{\partial V}{\partial z}\right)^2 + 2\left(\frac{U}{r}\right)^2 + \left[\left(\frac{\partial U}{\partial z}\right) + \left(\frac{\partial V}{\partial r}\right)\right]^2 \tag{72}$$

and

$$(\Omega\Omega')^2 = \frac{1}{4}\left[\left(\frac{\partial \Omega^2}{\partial r}\right)^2 + \left(\frac{\partial \Omega^2}{\partial z}\right)^2\right]. \tag{73}$$

These, and all other quantities, may be non-dimensionalized in the manner described above, and the heuristic model applied. Numerical integration indicates that λ_1 is roughly the same as s, the jet width parameter; this seems reasonable. λ_2 (which we will take as λ) remains nearly constant at about 0.1 across most of the jet and declines only slowly toward zero at relatively large distances from the center-

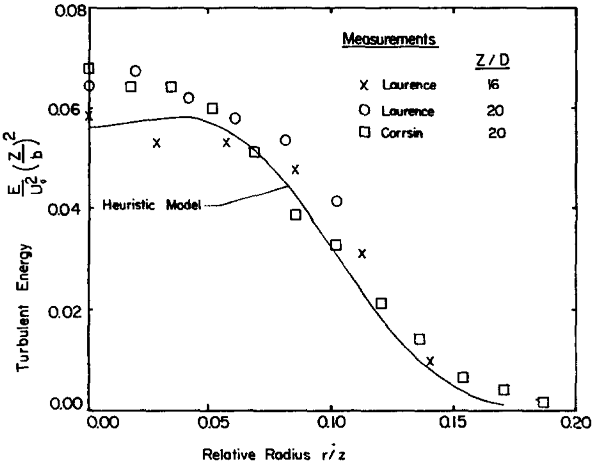

FIG. 6. Turbulent energy distribution in a circular jet.

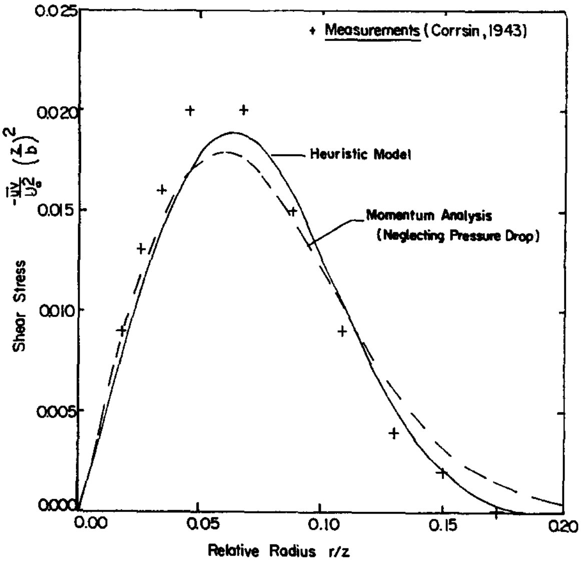

FIG. 7. Reynolds shear stress distribution in a circular jet.

line. The energy equation may now be integrated numerically subject to the boundary conditions

$$\partial E/\partial\eta = 0 \qquad \text{at} \qquad \eta = 0 \tag{74}$$

$$E \to 0 \qquad \text{as} \qquad \eta \to \infty. \tag{75}$$

The boundary condition (75) at infinitely must be satisfied indirectly. Actually, the integration starts at the centerline using (74) and a trial value of $E(0)$. A value of $E(0)$ is found by iteration which yields $E = 0$ at some large value η. In this regard the value $\eta = 0.25$ may be regarded as large. It is found that the precise position of the outer boundary makes virtually no difference in the results as long as it is greater than about 0.20.

The energy distribution found in this way is shown by the solid line in Fig. 6. The agreement with the experimental results of Corrsin [7] and of Laurence [8] is satisfactory, especially in view of the degree of scatter in the data points.

Once the energy distribution is known the eddy viscosity is fixed by Eq. (32) and the corresponding shear stresses by Eq. (71). These last are shown by the solid line in Fig. 7. The dashed line shows the corresponding shear stresses as computed from the momentum equation, as discussed earlier. Data points are from Corrsin [9].

If the heuristic model were completely correct, and if the assumed Gaussian velocity profile were exact, then the two shear stress curves shown in Fig. 7 would coincide exactly. As it is, the degree of agreement attained is considered to be reasonably satisfactory. The discrepancy between the two curves is smaller than the discrepancy between the experimental points and either of the curves. It seems probable that in this instance the theoretically computed values are actually more accurate than the experimentally measured ones; the experimental measurement is innately difficult and uncertain.

It may be concluded that, on the whole, these results for the turbulent jet substantiate quite well the proposed heuristic model of fluid turbulence.

Conclusions and Recommendations

It is concluded, on the basis of the evidence available so far, that the proposed [illegible] tics in the general case of inhomogeneous and nonstationary turbulence in incompressible flow.

It is recommended that the present model be applied also to other cases including pipe flow, two and three dimensional wakes, boundary layers, and to some examples of unsteady flow. The aim would be to refine the present unified theory and extend its range of applicability.

It is also recommended that further experimental information be obtained in connection with those aspects of the model for which the present data are insufficient. These aspects include, for example,

(a) the relation between the local length scale λ of the mean flow and the correlation length λ^* of the turbulence;

(b) the generalized three dimensional stress/strain relations which actually exist in regions of strong anisotropy such as in the flow near a wall; and

(c) the influence of various key parameters on the rate of dissipation of turbulent energy to heat.

ACKNOWLEDGMENTS

The authors express their appreciation to Dr. S. J. Lukasik, Deputy Director of the Advanced Research Projects Agency, Department of Defense, for the interest and support which made this research possible.

This work was supported jointly by the Advanced Research Projects Agency (Program Code 8F40, Order No. 961) and the Naval Ship Systems Command (Subproject SR 104 03 01).

REFERENCES

1. L. PRANDTL AND K. WIEGHARDT, Über ein neues Formelsystem fur die ausgebildete Turbulenz, *Nachr. Akad. Wiss. Goettingen* **19** (1945), 6.
2. F. H. HARLOW AND P. I. NAKAYAMA, Turbulence transport equations, *Phys. Fluids* **10** (1967), 2323.
3. C. W. HIRT, "Computer Studies of Time-Dependent Turbulent Flows," presented at the International Symposium on High-Speed Computing in Fluid Dynamics, Monterey, California, 1968.
4. J. LAUFER, "The Structure of Turbulence in Fully Developed Pipe Flow," NACA Report No. 1174 (1952).
5. H. REICHARDT, Messungen turbulenter Schwankungen, *Naturwissenschaften* **26** (1938), 404.
6. J. LAUFER, "Investigation of Turbulent Flow in a Two-Dimensional Channel," NACA Report No. 1053 (1949).
7. S. CORRSIN AND M. S. UBEROI, "Further Experiments on the Flow and Heat Transfer in a Heated Turbulent Air Jet," NACA TN 1865 (1947).
8. J. C. LAURENCE, "Intensity, Scale, and Spectra of Turbulence in Mixing Region of Free Subsonic Jet," NACA Report No. 1292 (1956).
9. S. CORRSIN, "Investigation of Flow in an Axially Symmetrical Heated Jet of Air," NACA WR W-94 (1943).